An Apparent Anomaly in the Group and Energy Velocity in a Dielectrically Loaded Slow-Wave Structure*

G. B. WALKER†, AND C. G. ENGLEFIELD†, MEMBER, IRE

Summary—This paper is concerned with group and energy velocity in a cylindrical guide periodically loaded with dielectric disks. For a TM wave a confluence may be obtained between the first and second pass band of such a structure by arranging that the characteristic impedances of the air and dielectric regions are equal when the phase change per section is π.

It was at first thought that, since the impedances are equal, there would be no reflections from the interfaces. Assuming only forward waves, however, the equivalence of group and energy velocity is violated.

The detailed analysis presented here shows that, mathematically, infinitely many solutions for the field pattern at the matched π mode are possible but that only one of these has physical significance. For this one pattern the group and energy velocities are equal.

Introduction

It IS well known that slow-wave structures, as used in electron tubes and wave filters, are inherently dispersive, that is, the velocity of propagation of phase is dependent upon frequency.[1] In discussing the performance of such structures a knowledge of this dependence is essential.

Wave propagation in periodic structures is characterized by the fact that the total wave can be regarded as a system of space harmonics, or component waves, each having a different phase velocity. The phase propagation constants of the components are related in a simple way, differing from each other by an integral multiple of $2\pi/L$ where L is the periodic length. Thus if β_n is the propagation constant of the nth space harmonic

$$\beta_n = \beta_0 + \frac{2\pi n}{L},$$

where β_0 is the propagation constant of a reference space harmonic.

Since space harmonics cannot exist independently, it is improper to refer to a phase velocity when describing the total wave disturbance. It may be noted, however, that the quantity $d\beta/d\omega$ is the same for all space harmonics and hence may be used to describe the dispersive character of the structure.

The term group velocity has been given to the quantity $d\omega/d\beta$ and it is customary to refer to the group velocity of wave propagation in a periodic structure in this way. For the purpose of this paper no other significance is attached to the term group velocity, and the phrase is used in this limited sense only.

The historical use of the term group velocity relates to the propagation of a wave packet; that is, a group of waves contained within a narrow frequency band. If $d\omega/d\beta$ is not a constant over this frequency band, a degeneration of the wave packet will result and it becomes difficult to ascribe a velocity to the packet.[2]

A quantity of physical importance is the velocity of propagation of energy along the structure. This also requires careful definition. By analogy with fluid flow, energy velocity may be defined as the ratio energy flux per unit area and per unit time, divided by the energy density. In the case of a periodically loaded waveguide, time average values of these quantities are used and the energy velocity is taken as the ratio of the time average power flux across any section of the guide to the mean stored energy per unit length of the guide, that is,

$$v_E = \frac{\int \frac{1}{2} \operatorname{Re} E \times H^* \cdot dS}{\frac{1}{L} \int \frac{1}{4} (\epsilon E E^* + \mu H H^*) dv},$$

where the integration in the numerator is over a section of the guide and the integration in the denominator is over the volume of one period.

An important theorem, for which proofs have been given by J. S. Bell[3] and D. A. Watkins,[4] states that in any lossless periodically loaded waveguide the group velocity and the energy velocity are equal. This theorem is only applicable when the loading obstacles consist of either metal or nondispersive dielectric materials. The case of dispersive media has recently been discussed by Tonning.[5]

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† Dept. of Elec. Engrg., University of British Columbia, Vancouver, Canada.

[1] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 5, sect. 5.14; 1941.

[2] R. B. Adler, *et al.*, "Electromagnetic Energy Transmission and Radiation," John Wiley and Sons, Inc., ch. 5, sec. 2.2; 1960.

[3] J. S. Bell, "Group Velocity and Energy Velocity in Periodic Waveguides," Atomic Energy Research Establishment, Harwell, England, AERE Rept. T/R 858; 1952.

[4] D. A. Watkins, "Topics in Electromagnetic Theory," John Wiley and Sons, Inc., New York, N. Y., ch. 1, sec. 5; 1958.

[5] A. Tonning, "Energy density in continuous EM media," IRE Trans. on Antennas and Propagation, vol. AP-8, pp. 428–434; July, 1960.

An apparent anomaly in the equivalence of group and energy velocity was noted in recent work by the authors in relation to waveguides loaded with disks of nondispersive dielectric material. The resolution of the anomaly has an important bearing on the understanding of pass-band confluence and is discussed here in some detail.

The Matched π Mode

In the dielectrically loaded structure shown in Fig. 1, for the TM_{01} mode, a confluence may be obtained between the first and second pass bands.

At a frequency f_m, given by

$$f_m = f_c\sqrt{\frac{\epsilon_r + 1}{\epsilon_r}},$$

where f_c = the air cutoff frequency,

and

$$\epsilon_r = \epsilon_2/\epsilon_1,$$

the characteristic impedance of the dielectric and air regions will be equal.[6] If the dimensions are chosen such that the phase change per section ψ is equal to π at the matching frequency f_m, then confluence of the pass bands will be obtained. Match can only occur at one frequency so that only one pair of pass bands can be made confluent.

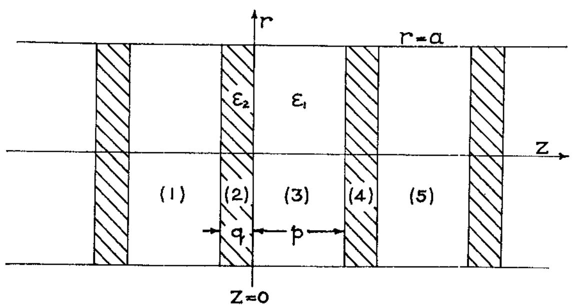

Fig. 1—Circular guide periodically loaded with dielectric disks.

Since the impedances of the two regions are equal, it would appear that a wave may travel down the structure with no reflections occurring at the interfaces. This being so there are no backward waves and the energy velocity in either region is the energy velocity for a single wave in that medium. It happens that for both regions $v_E = c/\sqrt{1+\epsilon_r}$, so that this is also the value for the over-all structure. To obtain the group velocity the determinantal equation has to be differentiated twice (l'Hôpital's rule); the value obtained is not equal to v_E. For $\epsilon_r = 93.5$, $f_m = 2998$ Mc, and a periodic length of 5 cm ($=\lambda_m/2$) the energy velocity is 3.1×10^7 m/sec and the group velocity, 6.8×10^7 m/sec.

[6] G. B. Walker and N. D. West, "Mode separation at the π-mode in a dielectric loaded waveguide cavity," *Proc. IEE*, vol. 104, p. 381, 1957; Monograph No. 228R, March, 1957.

If the dimensions are such that $\psi \neq \pi$ at the matching frequency, then group and energy velocity both have the value $c/\sqrt{1+\epsilon_r}$.

The discrepancy, therefore, only occurs when the match condition and the π mode occur simultaneously.

Analysis of the General Structure

A circular guide loaded with dielectric disks is shown in Fig. 1, where the even-numbered regions are dielectric disks of permittivity ϵ_2 and the odd-numbered regions are air spaces. We may determine the field patterns for a TM_{01} mode in this structure by matching fields at the boundaries of any one region.

The waves that may be set up in the air and dielectric regions may be found using Maxwell's equations, and for the TM_{01} mode are

$$\left.\begin{aligned} E_z &= (Ae^{-j\beta_2 z} + Be^{+j\beta_2 z})J_0(\chi r)\\ E_r &= \frac{j\beta_2}{\chi}(Ae^{-j\beta_2 z} - Be^{+j\beta_2 z})J_1(\chi r)\\ H_\phi &= \frac{j\omega\epsilon_2}{\chi}(Ae^{-j\beta_2 z} + Be^{+j\beta_2 z})J_1(\chi r)\end{aligned}\right\} -q \le z \le 0. \quad (1)$$

In the dielectric region (2), and

$$\left.\begin{aligned} E_z &= (Ce^{-j\beta_1 z} + De^{+j\beta_1 z})J_0(\chi r)\\ E_r &= \frac{j\beta_1}{\chi}(Ce^{-j\beta_1 z} - De^{+j\beta_1 z})J_1(\chi r)\\ H_\phi &= \frac{j\omega\epsilon_1}{\chi}(Ce^{-j\beta_1 z} + De^{+j\beta_1 z})J_1(\chi r)\end{aligned}\right\} 0 \le z \le p \quad (2)$$

in the air region (3), where

$$\begin{aligned}\beta^2 &= \omega^2\mu\epsilon - \chi^2,\\ \chi &= S_1/a,\end{aligned} \quad (3)$$

where S_1 is the first root of $J_0(p) = 0$, and A, B, C, and D are related constants.

Using Floquet's theorem the field in region (4) is given by (1) multiplied by $e^{-j\psi}$, where ψ is the phase change per section to be determined.

Matching fields at $z=0$ and $z=p$ we obtain,

$$\begin{aligned}\beta_2A - \beta_2B - \beta_1C + \beta_1D &= 0\\ \epsilon_2A + \epsilon_2B - \epsilon_1C - \epsilon_1D &= 0\\ \beta_2Ae^{j(2\theta_2-\psi)} - \beta_2Be^{-j(2\theta_2+\psi)} - \beta_1Ce^{-2j\theta_1} + \beta_1De^{2j\theta_1} &= 0\\ \epsilon_2Ae^{j(2\theta_2-\psi)} + \epsilon_2Be^{-j(2\theta_2+\psi)} - \epsilon_1Ce^{-2j\theta_1} - \epsilon_1De^{2j\theta_1} &= 0 \quad (4)\end{aligned}$$

where $2\theta_1$ is the phase change in the air region, β_1p, and $2\theta_2$ is the phase change in the dielectric region, β_2q. For these equations to have a unique solution,

$$\begin{vmatrix} \beta_2 & -\beta_2 & -\beta_1 & \beta_1\\ \epsilon_2 & \epsilon_2 & -\epsilon_1 & -\epsilon_1\\ \beta_2e^{j(2\theta_2-\psi)} & -\beta_2e^{-j(2\theta_2+\psi)} & -\beta_1e^{-2j\theta_1} & +\beta_1e^{2j\theta_1}\\ \epsilon_2e^{j(2\theta_2-\psi)} & \epsilon_2e^{-j(2\theta_2+\psi)} & -\epsilon_1e^{-2j\theta_1} & -\epsilon_1e^{2j\theta_1}\end{vmatrix} = 0. \quad (5)$$

Expanding and using the fact that the wave impedance Z is, for a TM mode, equal to $\beta/\omega\epsilon$, we obtain[7]

$$\cos\psi = \cos 2\theta_1 \cos 2\theta_2 - \frac{1}{2}\left[\frac{Z_1}{Z_2} + \frac{Z_2}{Z_1}\right] \sin 2\theta_1 \sin 2\theta_2. \quad (6)$$

We can find the relative values of A, B, C, and D by taking the first three equations of (4),

$$\begin{bmatrix} \beta_2 & -\beta_2 & -\beta_1 \\ \epsilon_2 & \epsilon_2 & -\epsilon_1 \\ \beta_2 e^{j(2\theta_2-\psi)} & -\beta_2 e^{-j(2\theta_2+\psi)} & -\beta_1 e^{-2j\theta_1} \end{bmatrix} \begin{bmatrix} A/D \\ B/D \\ C/D \end{bmatrix} = \begin{bmatrix} -\beta_1 \\ +\epsilon_1 \\ -\beta_1 e^{2j\theta_1} \end{bmatrix}. \quad (7)$$

Simplifying these equations, we obtain

$$\frac{A}{D} = \frac{\epsilon_1 e^{j\psi}}{\epsilon_2} \cdot \frac{\cos 2\theta_1 + j\dfrac{Z_1}{Z_2}\sin 2\theta_1 - e^{-j(2\theta_2+\psi)}}{e^{j(\psi-2\theta_1)} - \cos 2\theta_2 - j\dfrac{Z_2}{Z_1}\sin 2\theta_2} \quad (8)$$

$$\frac{B}{D} = \frac{\epsilon_1 e^{j\psi}}{\epsilon_2} \cdot \frac{\cos 2\theta_1 - j\dfrac{Z_1}{Z_2}\sin 2\theta_1 - e^{j(2\theta_2-\psi)}}{e^{j(\psi-2\theta_1)} - \cos 2\theta_2 - j\dfrac{Z_2}{Z_1}\sin 2\theta_2} \quad (9)$$

$$\frac{C}{D} = \frac{e^{j(\psi+2\theta_1)} + j\dfrac{Z_2}{Z_1}\sin 2\theta_2 - \cos 2\theta_2}{e^{j(\psi-2\theta_1)} - j\dfrac{Z_2}{Z_1}\sin 2\theta_2 - \cos 2\theta_2}. \quad (10)$$

Solving (6) for any particular structure, over a range of frequency, shows that a dielectrically loaded structure acts like a band-pass filter, the nonpropagating or stop bands occurring when $|\cos\psi| > 1$.

Figs. 2 and 3 show various dispersion curves using the dimensionless parameters a/λ, p/a, and q/a. By this means the variable a is effectively eliminated. The lowest value of a/λ at which propagation will occur, is approximately constant for a given value of p/q, provided that $p+q/a$ is small compared to λ/a.

Condition for Match

For any transverse magnetic wave there is a frequency at which the characteristic impedances of the air and dielectrically filled guide are equal.

As stated earlier, this frequency is given by

$$f_m = f_c\sqrt{\frac{\epsilon_r + 1}{\epsilon_r}}. \quad (11)$$

[7] Below air cutoff β_1 becomes

$$\beta_1' = \sqrt{\chi^2 - \omega^2\mu\epsilon_1}.$$

Eq. (6) may be applied if β_1 and Z_1 are replaced by $-j\beta_1'$ and $-j\beta_1'/\omega\epsilon_1$, respectively.

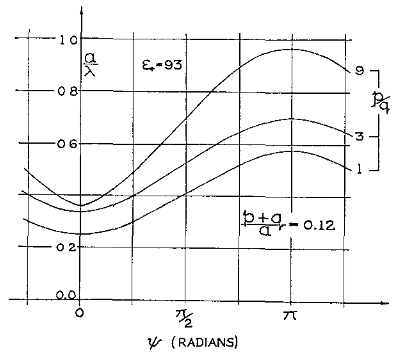

Fig. 2—Dispersion curves for various ratios of air to dielectric thickness.

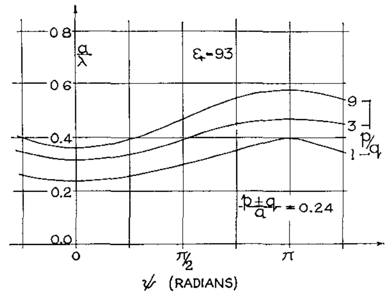

Fig. 3—Dispersion curves for various ratios of air to dielectric thickness.

At this frequency $\beta_2 = \epsilon_r\beta_1$ and $Z_1 = Z_2$ so that from (6), $\psi = 2\theta_1 + 2\theta_2$. We expect no reflections in the system and in general, this is true, there being no backward waves B and D. From (8) and (10) the forward waves are related simply by $C = \epsilon_r A$. Since there is only a single forward wave in either region, it is well known that both v_G and v_E are equal to $d\omega/d\beta$. For the air region

$$v_E = v_G = d\omega/d\beta_1 = \beta_1/\omega\mu\epsilon_1. \quad (12)$$

For the dielectric region

$$v_E = v_G = d\omega/d\beta_2 = \beta_2/\omega\mu\epsilon_2. \quad (13)$$

Since $\beta_2 = \epsilon_r\beta_1$, the energy velocities in the two regions are equal, and using (3) and (11), are given by

$$v_E = \frac{c}{\sqrt{1 + \epsilon_r}}. \quad (14)$$

Differentiation of (6) shows that the group velocity for the composite structure is also equal to this value.

The Matched π Mode

If we assume only A and C to exist at the matched π mode the energy velocity is given by (14), as above. The group velocity however cannot be found by a single differentiation of of (6) since both numerator and denominator vanish for this case. l'Hôpital's rule, however, may be used to obtain

$$v_G = \sqrt{\frac{c^2}{(\epsilon_r + 1) - \dfrac{c^2 \sin^2 2\theta_1}{\omega^2 (p+q)^2}\left(\epsilon_r - \dfrac{1}{\epsilon_r}\right)^2}}$$

$$\begin{bmatrix} Z_1 = Z_2 \\ \psi = \pi \end{bmatrix} \quad v_G = \frac{c}{\sqrt{\epsilon_r + 1 - \left[\dfrac{\lambda}{2\pi}\,\dfrac{\sin 2\theta_1}{p+q}\left(\epsilon_r - \dfrac{1}{\epsilon_r}\right)\right]^2}}\,. \qquad (15)$$

Eq. (15) gives a value of v_G greater than the value of v_E by (14) due to the additional term in the denominator. This appears to contradict the proofs of Bell and Watkins.

We can, however, view the problem in another way. At the π point, (10) is indeterminate; it is, in fact, possible to satisfy the boundary conditions with only forward waves, only backward waves, or any combination of these waves, *i.e.*, for any value of C/D. The forward waves and the backward waves do not interact as regards power flow, and from this point of view may be regarded as distinct. The mean power flow will be positive provided that $C>D$. The stored energy per section, however, depends on all four waves taken together. The energy velocity, being dependent on the ratio of these quantities, may thus have many values at the π mode.

The ratio C/D may thus be chosen such that D does not equal zero, C/D having a finite value. Calculations for a particular structure near to the π point, just above and just below the matching frequency, shows that C/D is continuous as the frequency changes through the π point.

The value of C/D at the π point may be found by differentiating numerator and denominator of (10) with respect to ω (l'Hôpital's rule), giving

$$-\frac{C}{D} = \frac{\dfrac{\sin 2\theta_1}{\omega}\cdot\dfrac{\epsilon_r^2 - 1}{\epsilon_r} + \left[\dfrac{(p+q)\sqrt{\epsilon_r + 1}}{c} + \dfrac{d\psi}{d\omega}\right] e^{2j\theta_1}}{\dfrac{\sin 2\theta_1}{\omega}\cdot\dfrac{\epsilon_r^2 - 1}{\epsilon_r} + \left[\dfrac{(p+q)\sqrt{\epsilon_r + 1}}{c} - \dfrac{d\psi}{d\omega}\right] e^{-2j\theta_1}}$$

$$\begin{bmatrix} \psi = \pi \\ Z_1 = Z_2 \end{bmatrix} \qquad (16)$$

where $d\psi/d\omega$ can be found from (15) since $d\psi/d\omega = v_G/(p+q)$.

The value of C/D, for a structure using titanium disks, is plotted against phase change per section in Fig. 4. C/D is infinite when $2\theta_2$ is equal to π, since for this condition each disk behaves as a resonant window and is reflectionless. The curve does not extend to the lower values of ψ, because (10) applies only above the

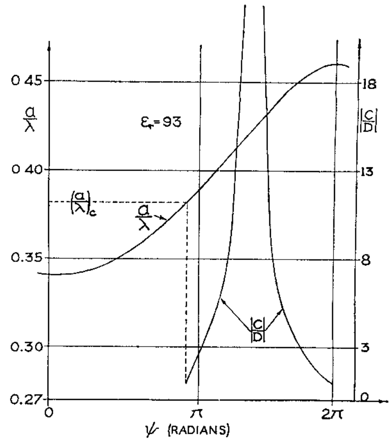

Fig. 4—Dispersion curve and variation of $|C/D|$, for a structure matched at π mode. ($f_m = 2998$ Mc, $p/a = 1.1766$, $q/a = 0.1228$, $a = 3.8479$ cm, $\epsilon_r = 93$.)

cutoff frequency, $(a/\lambda)_c$, for the air region. The value of C/D given by (16) agrees closely with that predicted from calculation at nearby points.

The energy velocity using this value of C/D, with the appropriate values of A/D and B/D, agrees closely with the group velocity (15).

Conclusions

For any system where the disks are matched but ψ does not equal π, the group velocity and energy velocity are equal. There are no backward waves, and this is the only possible solution.

If the disks are matched at the mathematically unique π-mode frequency, any combination of forward and backward waves is possible. From physical considerations it might be thought that, again, the condition of match implies no backward waves. Were this to be so, any change in frequency, however small, would cause an entirely different field pattern to be set up. The equivalence proofs of Bell and Watkins both require that $\partial F/\partial\omega$ is continuous at the point considered, F being any field vector.

It may be concluded, therefore, that the pattern which in fact will exist at the matched π-mode frequency, is that for which the ratio of the amplitudes of the forward and backward waves suffer no discontinuity with a small change in frequency [see (16)]. This is the only case for which $d\omega/d\beta$ has a meaning; the concept of group velocity may be applied only in this instance.

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